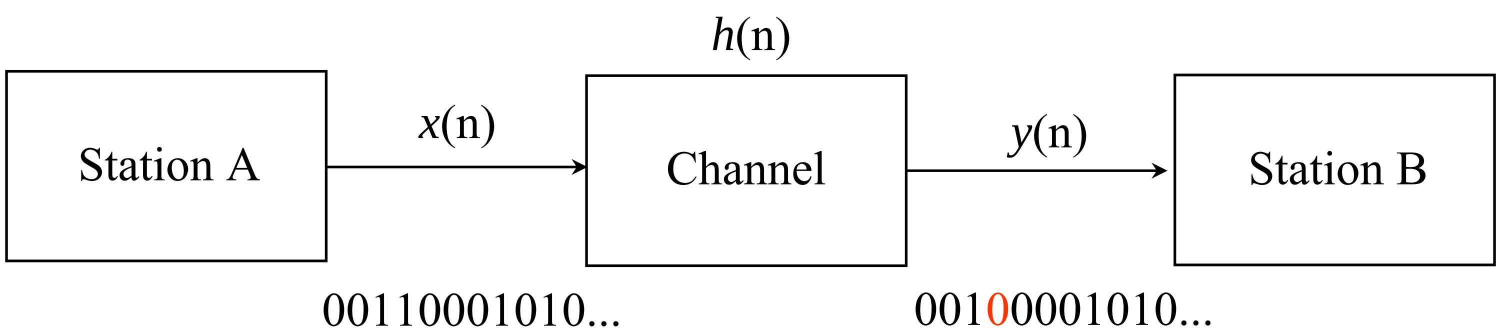
Typical Electrical Engineering Problems

A noisy binary communication channel

- The channel can be twisted pair, coaxial cable, fiber optic cable, or wireless medium.
- The channel introduces noise and thereby bit errors.



Typical Electrical Engineering Problems

Signal detection

- Desired target signal is buried in noise.

$$x(t) = A(t)\cos(\omega t + \phi(t)) + n(t)$$

- Determine the presence or absence of the desired signal.
- Filter the signal out of noise.
- Demodulate the signal.

Typical Electrical Engineering Problems

Networks

- In large computer networks, there are limited resources (e.g., bandwidth, routers, switches, printers and other devices) that need to be shared by the users.
  - User jobs/packets are queued and assigned service based on predefined criteria.
  - Demand is uncertain and service time is also uncertain.
  - Delay from the time the service is requested to the time it is completed.
- Telephone networks, multiuser computer networks, and other communication networks.

Our Interest and Goals

- Study tools to characterize the uncertainty

    - Probability theory, random variables, random processes

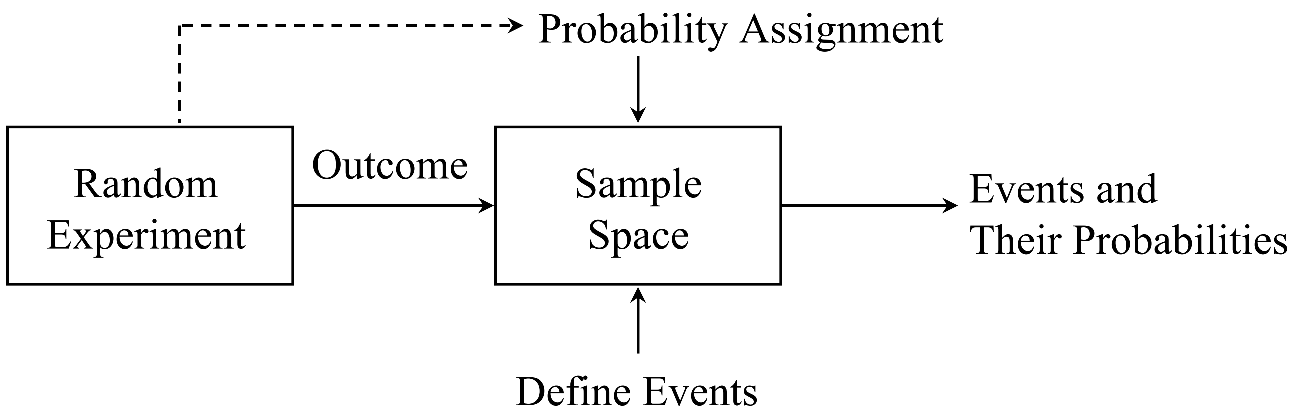
- Apply the tools to characterize non-deterministic signals

    - Random events, random signals

- Analyze systems processing non-deterministic signals

    - LTI systems with random inputs
    - Communication channels with noise
    - Communication networks with uncertain delays

Basics of Probability



- In a random experiment, the outcome is uncertain
  - Physical experiment
  - Abstraction
- The entire collection of outcomes is the sample space, S
  - Universal event or certain event
- An event consists of a single or a group of outcomes.
  - Events are user defined: A, B, …
- A measure of likelihood of occurrence of an event, A
  - Probability of A or Pr[A]

Example:

Roll a die once. All faces are equally likely.

- Sample space [discrete sample space]

$$S = \{1,2,3,4,5,6\}$$

- Define events:

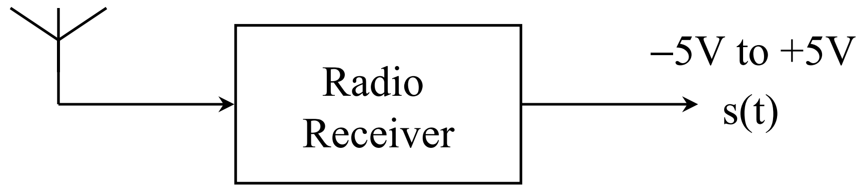
$$A_1 = \{\text{Odd numbered face}\} = \{1,3,5\}$$
$$A_2 = \{\text{Face value} < 3\} = \{1,2\}$$
$$A_3 = \{\text{Even numbered face value}\} = \{2,4,6\}$$

- Probability assignment:

$$\Pr[A_1] = 3/6 = 1/2$$
$$\Pr[A_2] = 2/6 = 1/3$$
$$\Pr[A_3] = 3/6 = 1/2$$

Continuous Sample Space

Continuous sample space <=> Continuous magnitude signals



- Output of the radio receiver is measured at $t = t_1$. The dynamic range of the receiver output is –5V to +5V

$$S = \{s: -5 \le s \le +5\}$$

- A continuous sample space has uncountably infinite values or outcomes
  - s could take values like 4.9326784531432677…
- Examples of events and probability assignments:

$$A_1 = \{s: \ -2.5 \le s \le 2.5\}, \quad \Pr[A_1] = 0.50$$

$$A_2 = \{s: \ -1 \le s \le 1\}, \quad \Pr[A_2] = 0.20$$

$$A_3 = \{s: \ s = 2.3 + \Delta x\}, \quad \lim_{\Delta x \to 0} \Pr[A_3] = 0$$

Discrete Sample Space

Discrete sample space <=> Discrete valued signals

- Output of an 8-bit ADC contains only $2^8 = 256$ values

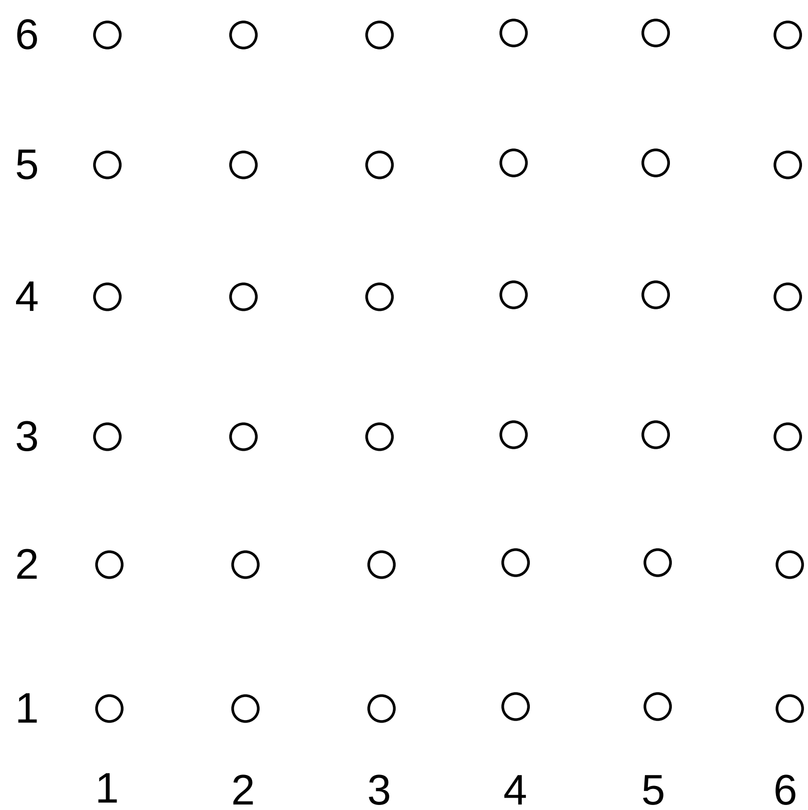
  S = {–5, –4.9609375, …, –0.0390625,0,0.0390625, …, 4.9609375} or
  S = {–128, –127, …, –1, 0, 1, …, 127}: decimal equivalent of 2’s complement representation

- A discrete sample space has finite or countably infinite values or outcomes
  - In this case, we have 256 values or outcomes (finite)

Two Dimensional Sample Space

Roll two dice [discrete sample space]

S = {(i,j): (1,1), (1,2), (1,3), …, (4,3), …, (6,4), (6,5), (6,6)}



Events and Event Operations

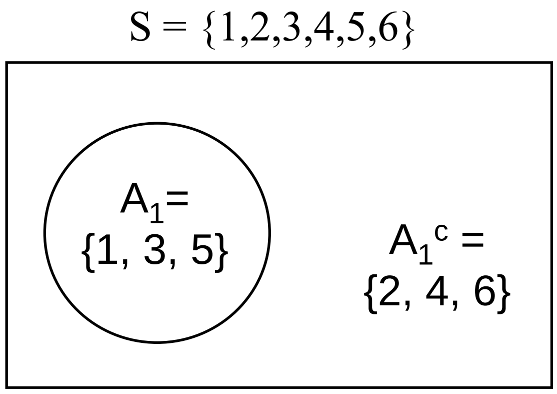
S	Certain event
0	Null event
$A_1, A_2, A_3, \ldots$	User defined events
$A_1 + A_2$	Union operation $(A_1 \cup A_2)$
$A_1 A_2$	Intersection operation $(A_1 \cap A_2)$
A^C	Complement operation

Example:

Consider a sample space represented by $S = \{1,2,3,4,5,6\}$

Let $A_1 = \{1,3,5\}$, $A_2 = \{1,2\}$, and $A_3 = \{2,4,6\}$ be the user defined events

- $A_1^C = (\{1,3,5\})^C = \{2,4,6\} = A_3$
- $A_2 + A_3 = \{1,2,4,6\}$
- $A_1 + A_3 = \{1,2,3,4,5,6\} = S$
- $A_1 A_2 = \{1\}$, $A_2 A_3 = \{2\}$
- $A_1 A_3 = 0$
- $A_1 + 0 = A_1$
- $A_1 0 = 0$



Postulates for the Algebra of Events

• $A_1 A_1^C = 0$	Mutual exclusion
• $A_1 S = A_1$	Inclusion
• $(A_1^C)^C = A_1$	Double complement
• $A_1 + A_2 = A_2 + A_1$	Commutative law
• $A_1 + (A_2 + A_3) = (A_1 + A_2) + A_3$	Associative law
• $A_1 (A_2 + A_3) = A_1 A_2 + A_1 A_3$	Distributive law
• $(A_1 A_2)^C = A_1^C + A_2^C$	De Morgan's law

Other Identities

- $S^C = 0$
- $A_1 + 0 = A_1$ Inclusion
- $A_1 A_2 = A_2 A_1$ Commutative law
- $A_1 (A_2 A_3) = (A_1 A_2) A_3$ Associative law
- $A_1 + (A_2 A_3) = (A_1 + A_2)(A_1 + A_3)$ Distributive law
- $(A_1 + A_2)^C = A_1^C A_2^C$ De Morgan's law

Finite Unions and Intersections

These are included in the algebra.

$$\bigcup_{i=1}^{N} A_i = A_1 + A_2 + A_3 + \ldots + A_N$$

$$\bigcap_{i=1}^{N} A_i = A_1 \ A_2 \ A_3 \ \ldots \ A_N$$

Infinite Unions and Intersections

If they are included, the algebra of events is called a Sigma Algebra.

$$\bigcup_{i=1}^{\infty} A_i = A_1 + A_2 + A_3 + \ldots$$

$$\bigcap_{i=1}^{\infty} A_i = A_1 \ A_2 \ A_3 \ \ldots$$

Mutually Exclusive and Collectively Exhaustive Sets of Events

Mutually exclusive: $A_k \; A_j = 0 \qquad k \neq j$

Collectively exhaustive: $\bigcup_j A_j = S$

Working Definition of the Sample Space

THE SAMPLE SPACE IS REPRESENTED BY THE FINEST GRAIN, MUTUALLY EXCLUSIVE, COLLECTIVELY EXHAUSTIVE SET OF OUTCOMES FOR AN EXPERIMENT.

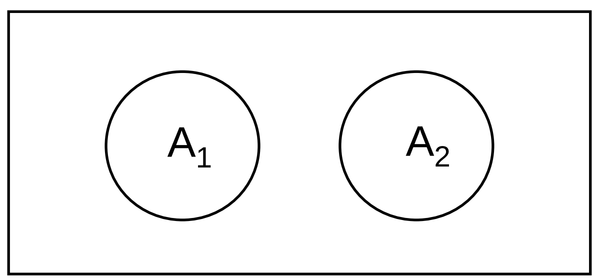
The Axioms of Probability

I. $\Pr[A_i] \geq 0$ for any event

II. $\Pr[S] = 1$

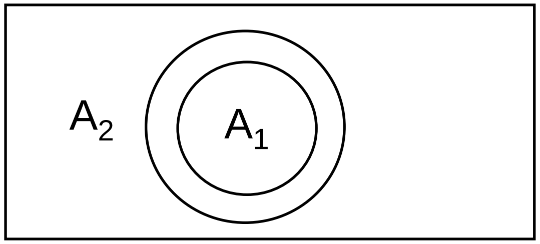
III(a). If $A_1 A_2 = 0$, then $\Pr[A_1 + A_2] = \Pr[A_1] + \Pr[A_2]$

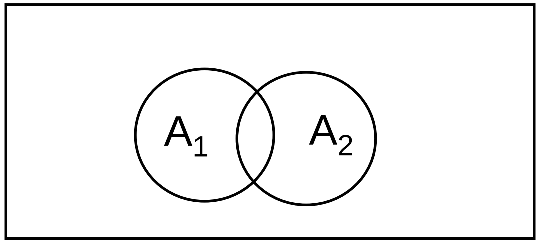
III(b). If $A_i A_j = 0$ for $i \neq j$, then $\Pr\left[\bigcup_{i=1}^{\infty} A_i\right] = \sum_{i=1}^{\infty} \Pr[A_i]$



Some Corollaries

1. $\Pr[A^c] = 1 - \Pr[A]$ (ii, iii)
2. $0 \leq \Pr[A] \leq 1$ (i, ii, iii)
3. If $A_1 \subset A_2$, then $\Pr[A_1] \leq \Pr[A_2]$ (i, iii)
4. $\Pr[0] = 0$ (ii, iii)
5. If $A_1 A_2 = 0$, then $\Pr[A_1 A_2] = 0$
6. $\Pr[A_1 + A_2] = \Pr[A_1] + \Pr[A_2] - \Pr[A_1 A_2]$





The Principle of Total Probability

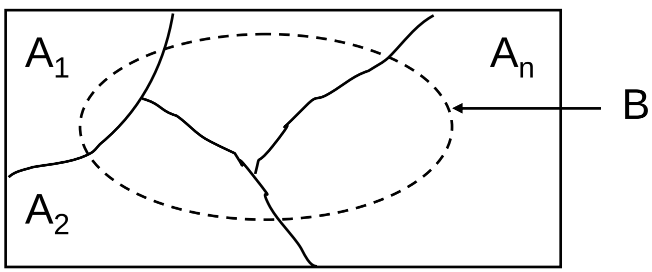
Let A_1, A_2, …, A_n be a set of mutually exclusive and collectively exhaustive events:

$$A_k A_j = 0 \qquad k \neq j$$

$$\bigcup_{j=1}^{n} A_j = S \quad \text{then} \quad \sum_{j=1}^{n} \Pr[A_j] = 1$$

Now let B be <u>any</u> event in S. Then,

$$\Pr[B] = \Pr[BA_1] + \Pr[BA_2] + \cdots + \Pr[BA_n]$$



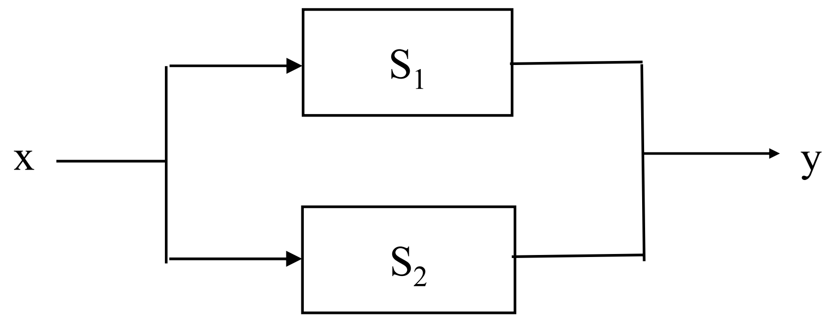
Independence of Events

Two events A_1 and A_2 are said to be statistically independent if and only if

$$\Pr[A_1A_2] = \Pr[A_1]\,\Pr[A_2]$$

System Reliability Calculations

Parallel Connection of switches



Define: $A_1 = \{S_1 \text{ fails}\}$, $\Pr[A_1] = p$, $\Pr[A_1^c] = 1 - p = q$

$A_2 = \{S_2 \text{ fails}\}$, $\Pr[A_2] = p$, $\Pr[A_2^c] = 1 - p = q$

$F = \{\text{no connection between x and y}\} = A_1 A_2$

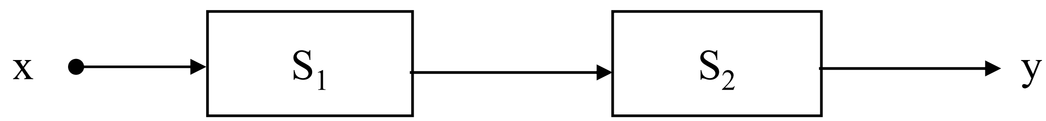
The probability that the connection fails:

$$\Pr[F] = \Pr[A_1 A_2] = \Pr[A_1]\Pr[A_2] \qquad A_1 \text{ and } A_2 \text{ are independent}$$

$$= p^2$$

System Reliability Calculations

Series connection of switches



Assume that switch failures are statistically independent; failure results in an open connection.

$$F = \{\text{no connection between x and y}\} = A_1 + A_2$$

The probability that the connection fails:

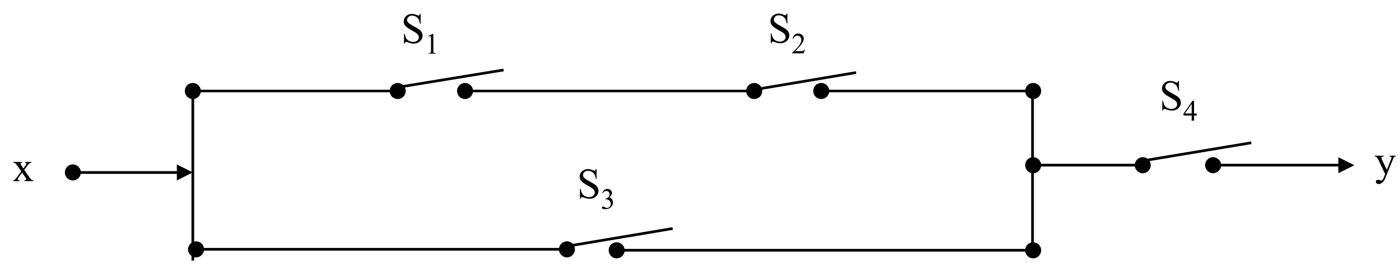
$$\Pr[F] = \Pr[A_1 + A_2] = \Pr[A_1] + \Pr[A_2] - \Pr[A_1 A_2]$$

$$= \Pr[A_1] + \Pr[A_2] - \Pr[A_1]\Pr[A_2] \qquad A_1 \text{ and } A_2 \text{ are independent}$$

$$= p + p - p^2 = 2p - p^2$$

Example:

Consider a simple switching network as follows:



Define: A_k = {switch S_k fails}, k = 1,2,3,4 $\Pr[A_k] = p$

(a) Find the probability that the path between x and y is established.

Let F = {no connection between x and y} $= (A_1 + A_2)A_3 + A_4$

The probability of path failure is given by

$$\Pr[F] = \Pr[(A_1 + A_2)A_3 + A_4] = \Pr[A_1A_3 + A_2A_3] + \Pr[A_4] - \Pr[A_1A_3A_4 + A_2A_3A_4]$$

$$= \Pr[A_4] + \Pr[A_1A_3] + \Pr[A_2A_3] - \Pr[A_1A_2A_3]$$

$$- \Pr[A_1A_3A_4] - \Pr[A_2A_3A_4] + \Pr[A_1A_2A_3A_4]$$

$$= p + 2p^2 - 3p^3 + p^4$$

The desired probability is then given by

$$\Pr[\text{path established}] = 1 - \Pr[F] = 1 - p - 2p^2 + 3p^3 - p^4$$

(b) Compute the desired probability as a function of p:

$$\Pr[\text{path established}] = 1 - \Pr[F] = 1 - p - 2p2 + 3p3 - p4$$

p	1 – Pr[F]
0.1	0.8829
0.01	0.98980299
0.001	0.998998002999
0.0001	0.999899980003

Repeated Independent Trials (Bernoulli Trials)

A random experiment, E, consists of several sub-experiments or trials, E_i:

- All sub-experiments have the same sample space, S_i.
- Events from all sub-experiments are mutually independent:

  $$\Pr[A_1 A_2 A_3 \ldots A_n] = \Pr[A_1]\Pr[A_2]\Pr[A_3]\ldots\Pr[A_n],$$

  where A_i is an event from S_i.
- The sample space of E is:

  $$S = S_1 \times S_2 \times S_3 \times \ldots \times S_n$$

Counting Methods and Probability

The assignment of probability is given by

$$\Pr[A_1] = \frac{\text{Number of outcomes in Event } A_1}{\text{Number of outcomes of experiment}}$$

The rule of products:

Consider an experiment with *n* outcomes; repeat *r* times

Total number of outcomes is given by

$$N_r^n = n \cdot n \cdot \cdots \cdot n = n^r$$

or in a general case

$$N_r^{n_i} = n_1 n_2 \ldots n_r = \prod_{i=1}^{r} n_i$$

Example:

Roll a die 4 times sequentially. The total number of outcomes is

$$N_4^6 = 6 \cdot 6 \cdot 6 \cdot 6 = 6^4 = 1296$$

Example:

Form 5-letter words using 26 English alphabet characters. Characters can be repeated, and the words so formed do not have to be meaningful.

$$N_5^{26} = 26^5 = 11{,}881{,}376$$

Example:

Construct variable names of length 3 using a letter, a number, and a letter (e.g., A2C).

$$N_3^{n_i} = 26 \cdot 10 \cdot 26$$

PERMUTATIONS (products without replacement)

Select r objects from among a given set of n distinct objects where we pay attention to the order in which the r objects are selected.

$$P_r^n = n(n-1)(n-2)\cdots(n-r+1)$$

$$= \frac{n!}{(n-r)!}, \qquad \text{for } r \leq n$$

Special case: For $r = n$: $\quad P_n^n = n!$

Example:

Form 5-letter words using the English alphabet. The characters cannot be repeated, and the words do not have to be meaningful.

The total number of words that can be formed is:

$$P_5^{26} = \frac{26!}{(26-5)!}$$

$$= \frac{26!}{21!} = 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 = 7{,}893{,}600$$

COMBINATIONS (without replacement, without order)

Select r objects from among a given set of n distinct objects where we <u>pay no attention to the order</u> in which the r objects are selected.

$$C_r^n = \frac{n(n-1)\cdots(n-r+1)}{r(r-1)(r-2)\cdots(1)} = \frac{n!}{r!(n-r)!} = \binom{n}{r}$$

Binomial coefficient "n *choose* r"

Example:

Consider 5 workstations having equal capabilities: $\{a, b, c, d, e\}$

- *Permutations*: Select two workstations where one will be a server and the other a graphics workstation. The possible selections are:

  ab ba ac ca ad da ae ea bc cb bd db be eb cd dc ce ec de ed

$$P_2^5 = \frac{5!}{(5-2)!} = 5 \cdot 4 = 20 \qquad \text{"ab} \neq \text{ba"}$$

- *Combinations*: Select two workstations where both will be used as graphics workstations. The possible selections are:

  ab ac ad ae bc bd be cd ce de

$$C_2^5 = \frac{5!}{2!(5-2)!} = \frac{5 \cdot 4}{2} = 10 \qquad \text{"ab = ba"}$$

Example:

We plan to buy 5 personal computers. The computer store has a stock of 10 foreign made PCs and 15 US made PCs that meet our specifications.

(a) Assuming that the 5 computers are randomly chosen from this lot, what is the probability that exactly 3 US made computers are selected?

- The sample space is given by

  S = {combinations of *r* = 5 chosen from *n* = 25}

$$N_S = C_5^{25} = \frac{25!}{5!(25-5)!}$$

- The desired event A = {*exactly* 3 of the 5 selected are US made}

$$N_A = C_3^{15} C_2^{10} = \frac{15!\,10!}{3!(15-3)!\ \ 2!(10-2)!}$$

- The probability that we have 3 US made computers is [Hyper geometric distribution]

$$\Pr[A] = \frac{N_A}{N_S} = \frac{C_3^{15} C_2^{10}}{C_5^{25}} = \frac{15!\ \ 10!\ \ 5!\ \ (25-5)!}{3!\ \ (15-3)!\ \ \ 2!\ \ (10-2)!\ \ \ 25!} \cong 0.3854$$

Example (continued):

(b) Assuming that the 5 computers are randomly chosen from this lot, what is the probability that *at least* 1 is foreign made?

- Let event B = {none of the 5 selected computers is foreign made}
- The desired event C = {one or more of the 5 are foreign made}
- Since $B^C = C$, we can write Pr[C] = 1 − Pr[B]

$$\Pr[B] = \frac{N_B}{N_S} = \frac{C_5^{15} C_0^{10}}{C_5^{25}} = \frac{15!\ \ 10!\ \ 5!\ \ (25-5)!}{5!\ \ (15-5)!\ \ 0!\ \ (10-0)!\ \ 25!} \cong 0.05652$$

$$\Pr[C] = 1 - \Pr[B] = 0.94348$$

Example:

Consider a box of 25 modem chips: 5 of them are known to be defective. Select 6 from the box at random and test them. What is the probability that exactly 2 are defective?

- Sample Space: S = {combinations of r = 6 chosen from n = 25}
- Event: A = {*exactly* 2 of the 6 selected chips are defective}
- The number of outcomes in S is given by:

$$N_S = C_6^{25} = \frac{25!}{6!(25-6)!}$$

- For the selected 6 chips, we are interested in the case, where 2 are defective (i.e., they are from the 5 defective chips in the box) and 4 are non-defective (i.e., they are from the 20 non-defective chips in the box).

Example (continued):

– The number of outcomes in A is given by:

$$N_A = C_2^5 C_4^{20} = \frac{5!\ 20!}{2!(5-2)!\ \ 4!(20-4)!}$$

– The probability that exactly 2 of the 6 selected chips are defective is [Hyper geometric distribution]

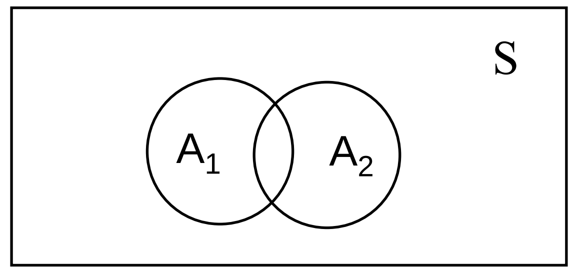
$$\Pr[\mathrm{A}] = \frac{N_A}{N_S} = \frac{C_2^5 C_4^{20}}{C_6^{25}}$$

$$= \frac{5!\ \ 20!\ \ 6!\ \ (25-6)!}{2!\ \ (5-2)!\ \ 4!\ \ (20-4)!\ \ 25!} \cong 0.2736.$$

Conditional Probability

Probability of occurrence of one event (say, A_1) subject to the knowledge that another event (say, A_2) has occurred.

$$\Pr\left[A_1 \middle| A_2\right] = \frac{\Pr[A_1 \quad A_2]}{\Pr[A_2]}$$



$\Pr[A_1 \mid A_2]$ is read as "probability of A_1 *given* A_2 "

If A_1 and A_2 are <u>independent</u>, then

$$\Pr\left[A_1 \middle| A_2\right] = \frac{\Pr[A_1 \quad A_2]}{\Pr[A_2]} = \frac{\Pr[A_1]\Pr[A_2]}{\Pr[A_2]} = \Pr[A_1]$$

<u>Example</u>:

Consider a sequence of 3 binary numbers (occurring randomly).

Sample space: S = {000, 001, 010, 011, 100, 101, 110, 111}

What is the probability that there are more 1's than 0's given that the first
bit is a 1.

- Let us define two events:

  A_1 = {more 1's than 0's} = {011, 101, 110, 111}

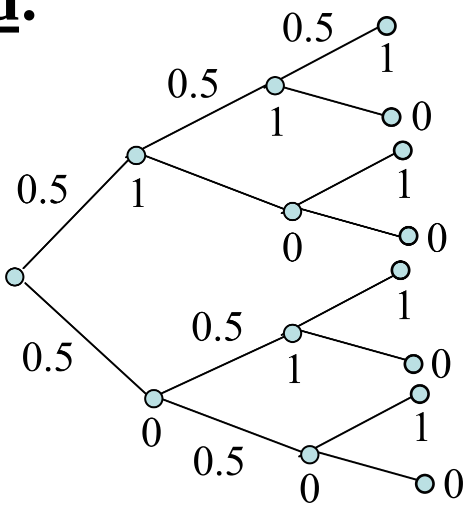
  A_2 = {the first bit is a 1} = {100, 101, 110, 111}

- Their intersection:

  A_1A_2 = {101, 110, 111}

Example cont'd:

Tree diagram



Sample Space
111
110
101
100
011
010
001
000

- All 8 events in the sample space have probability 1/8, therefore

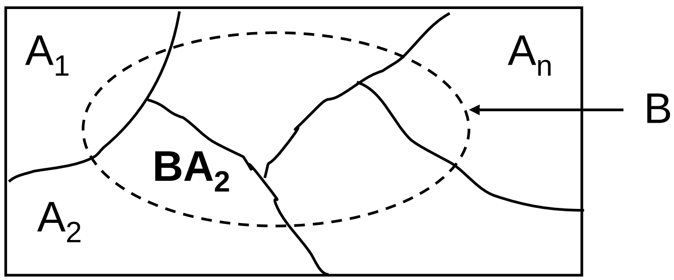
$$\Pr[A_2] = \frac{4}{8} \quad \text{and} \quad \Pr[A_1 A_2] = \frac{3}{8}$$

- The conditional probability is obtained as follows:

$$\Pr\left[A_1 \middle| A_2\right] = \frac{\Pr[A_1 \quad A_2]}{\Pr[A_2]} = \frac{3/8}{4/8} = \frac{3}{4}$$

The Principle of Total Probability Revisited

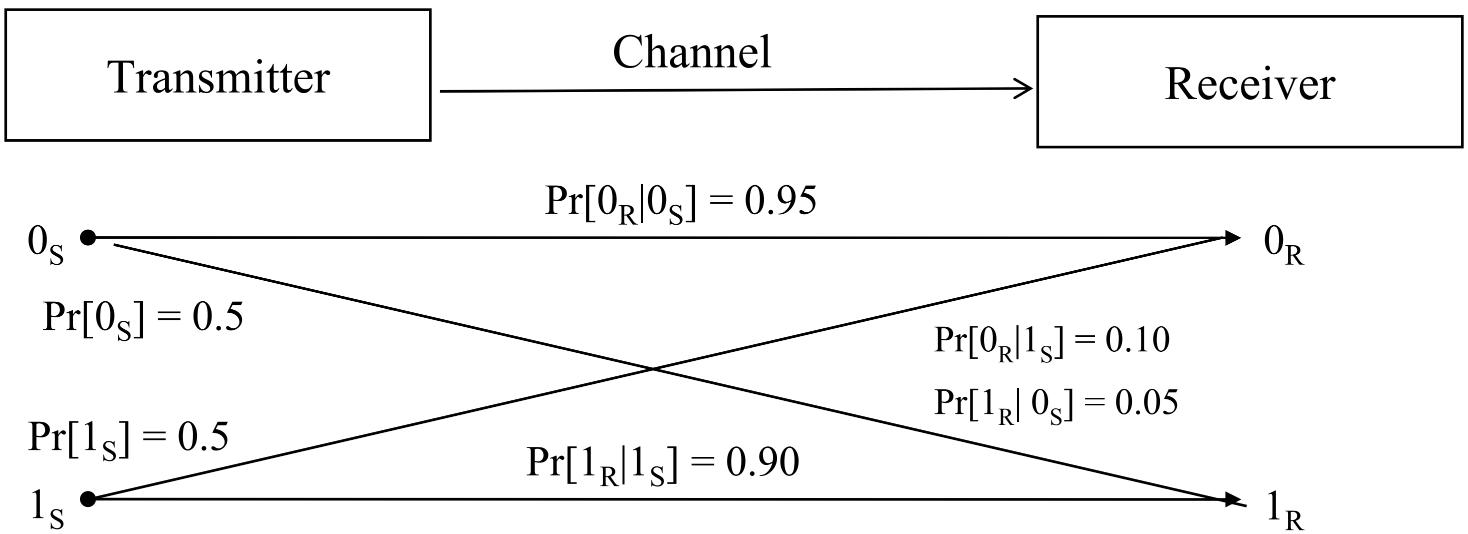
Let $A_1, A_2, \ldots, A_n$ be mutually exclusive and collectively exhaustive events. Let B be an event in S.



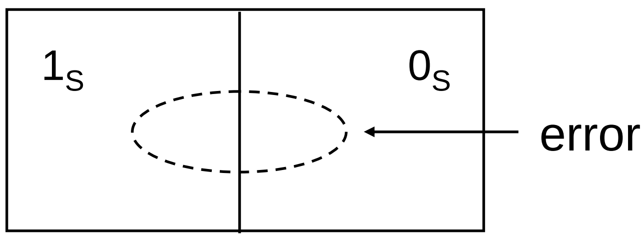
Then,

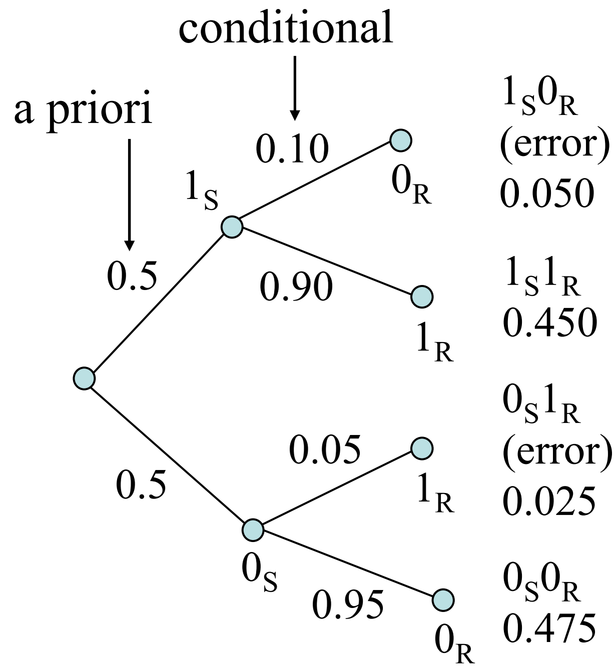
$$\begin{aligned}\Pr[B] &= \Pr[BA_1] + \Pr[BA_2] + \ldots + \Pr[BA_n] \\ &= \Pr\left[B \mid A_1\right] \Pr[A_1] + \ldots + \Pr\left[B \mid A_n\right] \Pr[A_n] \\ &= \sum_{i=1}^{n} \Pr\left[B \mid A_i\right] \Pr[A_i]\end{aligned}$$

<u>Example</u>: Binary Communication Channel



Example cont'd.





$$\Pr[\text{error} \mid 1_S] = \Pr[0_R \mid 1_S] = 0.10$$

$$\Pr[\text{error} \mid 0_S] = \Pr[1_R \mid 0_S] = 0.05$$

$$\begin{aligned}\Pr[\text{error}] &= \Pr[\text{error} \mid 1_S]\Pr[1_S] + \Pr[\text{error} \mid 0_S]\Pr[0_S] \\ &= 0.10 \cdot 0.50 + 0.05 \cdot 0.50 = 0.075\end{aligned}$$

More on Conditional Probability

From the definition of conditional probability, we can write

$$\Pr\left[A_1 \middle| A_2\right] = \frac{\Pr[A_1 \quad A_2]}{\Pr[A_2]} \qquad \text{or} \qquad \Pr[A_1 \quad A_2] = \Pr\left[A_1 \middle| A_2\right] \; \Pr[A_2]$$

$$\Pr\left[A_2 \middle| A_1\right] = \frac{\Pr[A_1 \quad A_2]}{\Pr[A_1]} \qquad \text{or} \qquad \Pr[A_1 \quad A_2] = \Pr\left[A_2 \middle| A_1\right] \; \Pr[A_1]$$

$$\therefore \quad \Pr\left[A_1 \middle| A_2\right] \; \Pr[A_2] = \Pr\left[A_2 \middle| A_1\right] \; \Pr[A_1]$$

This can be written as

$$\boxed{\Pr\left[A_2 \middle| A_1\right] = \frac{\Pr\left[A_1 \middle| A_2\right] \; \Pr[A_2]}{\Pr[A_1]}}$$

Bayes' Rule

Let A_1, A_2, …, A_n be a set of mutually exclusive, collective exhaustive events. Then,

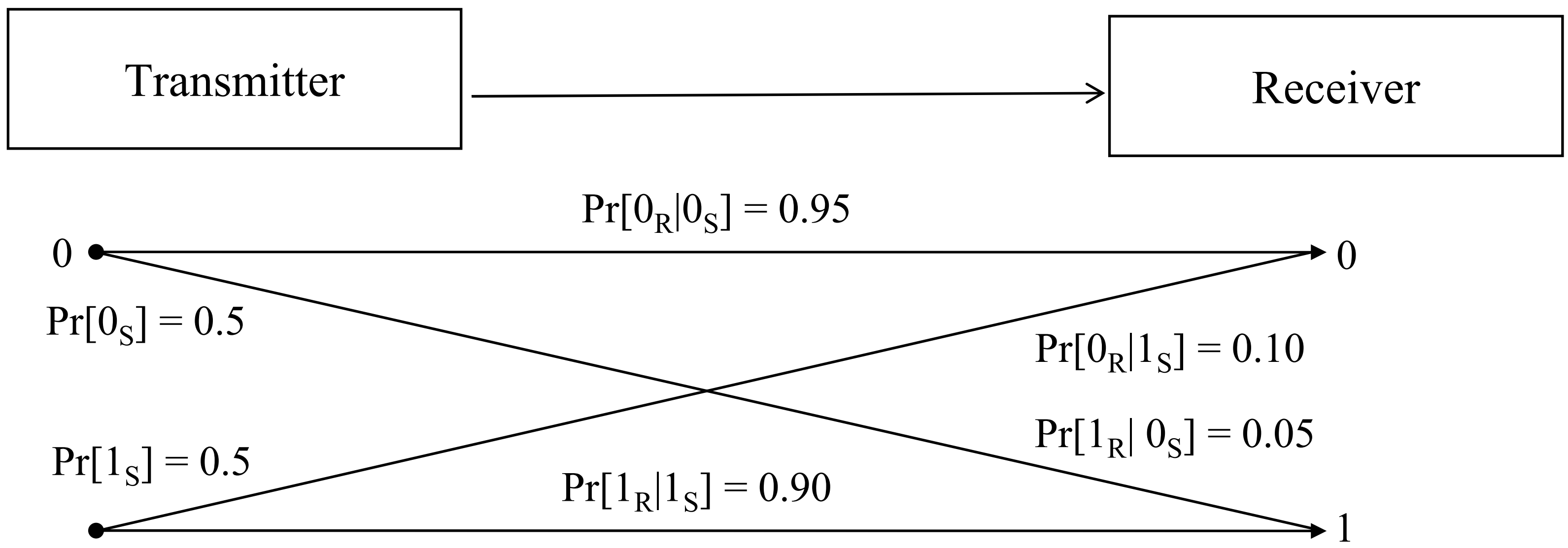
$$\Pr\left[A_j \mid B\right] = \frac{\Pr[B \mid A_j]\Pr[A_j]}{\Pr[B]}$$

Or, applying the Principle of Total Probability

$$\Pr\left[A_j \mid B\right] = \frac{\Pr[B \mid A_j] \;\; \Pr[A_j]}{\sum_{k=1}^{n} \Pr[B \mid A_k]\Pr[A_k]}$$

This is called Bayes' Rule.

Example: Binary Communication Channel



Determine the inverse probability, $P[1_S \,|1_R]$.

Example cont'd.

Determine the inverse probability, $P[1_S | 1_R]$:

$$\Pr[1_S | 1_R] = \frac{\Pr[1_R | 1_S]\Pr[1_S]}{\Pr[1_R]}$$

$$= \frac{\Pr[1_R | 1_S]\Pr[1_S]}{\Pr[1_R | 1_S]\Pr[1_S] + \Pr[1_R | 0_S]\Pr[0_S]}$$

$$= \frac{0.45}{0.45 + 0.025} = 0.9474$$

Example:

Consider 3 boxes of ICs:

Box 1 contains 1500 ICs and 10% of them are defective;

Box 2 contains 2000 ICs and 20% of them are defective; and

Box 3 contains 3000 ICs and 16% of them are defective.

Select 1 of the 3 boxes at random and choose an IC from that box at random.

(a) What is the probability that this IC is defective?

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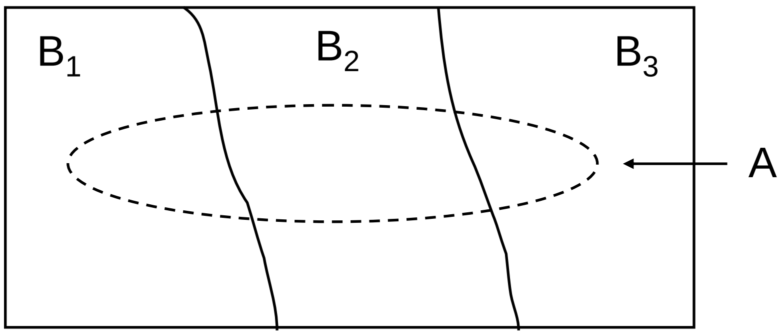

Define: A = “selected IC is defective” ,
B_i = “IC is from box i”

- By the principle of total probability, we can write

$$\Pr[A] = \Pr\left[A \middle| B_1\right]\Pr[B_1] + \Pr\left[A \middle| B_2\right]\Pr[B_2] + \Pr\left[A \middle| B_3\right]\Pr[B_3]$$

$$= 0.10 \cdot \frac{1}{3} + 0.20 \cdot \frac{1}{3} + 0.16 \cdot \frac{1}{3} = \frac{0.46}{3} = 0.1533$$

(b) Suppose that the selected IC is found to be defective. What is the probability that this IC came from box #3?

By Bayes' theorem, we can write

$$\Pr\left[B_3 \middle| A\right] = \frac{\Pr[A \mid B_3]\Pr[B_3]}{\Pr[A]}$$

$$= \frac{0.16 \cdot \frac{1}{3}}{\frac{0.46}{3}} = \frac{0.16}{0.46} = 0.3478$$

(c) Suppose all IC's are thoroughly mixed in one box and an IC is selected at random from the box. What is the probability that the IC is defective?

Basic Information Theory

Given an event A and its probability Pr[A], information associated with A is given by

$$I[\mathrm{A}] = \log_x \frac{1}{\Pr[\mathrm{A}]} = -\log_x \Pr[\mathrm{A}],$$

where x is the base of the logarithm:

if $x = 2$, the units of information are bits;

$x = 10$, the units are hartleys;

and $x = e$, the units are nats.

Note the identity: $\log_a b = x$ means that $a^x = b$.

Example:

Consider two events: A_1 and A_2 with corresponding probabilities of occurrence of 0.125 and 0.875, respectively.

The information associated with these events:

$$I[A_1] = -\log_2 (0.125) = 3 \text{ bits and } I[A_2] = -\log_2 (0.875) = 0.1925 \text{ bits.}$$

Entropy

Given a set of independent events that are mutually exclusive and collectively exhaustive, we can define the average information associated with the random experiment as

$$H = \sum_i \Pr[A_i] \cdot I[A_i] = -\sum_i \Pr[A_i] \cdot \log_x \Pr[A_i].$$

Example:

Consider a sequence 1 2 3 2 3 4 5 4 5 6 7 8 9 8 9 0.

We estimate the probability of occurrence of each symbol as follows:

$$\Pr[1] = \Pr[6] = \Pr[7] = \Pr[0] = 1/16$$

$$\Pr[2] = \Pr[3] = \Pr[4] = \Pr[5] = \Pr[8] = \Pr[9] = 2/16.$$

The entropy of this sequence is

$$H = -\sum_i \Pr[\mathrm{A}_i] \cdot \log_2 \Pr[\mathrm{A}_i] = 3.25 \quad \text{bits.}$$

Shannon-Fano Code

- Messages are composed of an alphabet in which the frequency of occurrence of each letter is a probabilistic phenomenon.
  - For transmission purposes the messages are compressed such that the code length of a letter is inversely proportional to its frequency of occurrence (e.g., think of the Morse code).
  - Since the letters are transmitted sequentially, no short codeword be part of the start of a longer codeword for unique decodability.
- Shannon-Fano Algorithm:
  - Arrange letters in a descending order of their probabilities by breaking any ties arbitrarily.
  - Starting at the top, partition the letters into two equi-probable subgroups (as closely as possible): assign 0 to the first subgroup and 1 to the second.
  - Continue partitioning the subgroups until all letters are exhausted: after each partition, assign a 0 to the first group and a 1 to the second and append the newly assigned bits to the previously assigned bits.

Example

Given a text message "ELECTRICAL ENGINEERING," determine the relative probabilities of the letters in the message and find the Shannon-Fano code for each letter. Ignore the space character.

- Since there are 21 letters in the message, we have the following probabilities:

  Letters: {E, L, C, T, R, I, A, N, G}

  Probabilities: {5/21, 2/21, 2/21, 1/21, 2/21, 3/21, 1/21, 3/21, 2/21}

- Code assignment:

						Codeword	Length
E	5/21	0	0			00	2
I	3/21	0	1	0		010	3
N	3/21	0	1	1		011	3
L	2/21	1	0	0		100	3
C	2/21	1	0	1	0	1010	4
R	2/21	1	0	1	1	1011	4
G	2/21	1	1	0		110	3
T	1/21	1	1	1	0	1110	4
A	1/21	1	1	1	1	1111	4

The Binomial Probability Law

Consider a sequence of *n* binary values:

$$\Pr[1] = p, \Pr[0] = 1 - p = q.$$

Define: A = {occurrence of r 1's in a sequence of length *n*}

The number of ways *r* 1's can occur in a sequence of length *n* is given by the binomial coefficient, C_r^n:

– Note that each of these arrangements has *r* 1's and $n - r$ 0's.

– The probability of occurrence of such an arrangement is then given
by $p^r q^{n-r}$.

The probability of the desired event: $\Pr[\mathrm{A}] = \binom{n}{r} p^r q^{n-r}$

Example 1:

Consider a modem connection with a channel bit error rate $p = 10^{-2}$. Given that the data are sent as packets of 100 bits, what is the probability that (a) 1 bit is in error and (b) 3 bits are in error?

$$\text{(a) } \Pr[1 \text{ bit in error}] = \binom{100}{1} 0.01^{1} \cdot 0.99^{99} = 0.3697$$

$$\text{(b) } \Pr[3 \text{ bits in error}] = \binom{100}{3} 0.01^{3} \cdot 0.99^{97} = 0.060999$$

Example 2:

Consider a communication system with a channel bit error rate, p = 10^{-3}. The transmitter sends each bit three times, and the receiver takes a majority poll of the received bits to determine the received bit. What is the probability of bit error now?

(a) Each transmission is a Bernoulli trial with n = 3.

Define: A = {2 or more bit errors in 3 trials}

$$\Pr[\text{error}] = \Pr[\text{A}] = \Pr[r \geq 2]$$

$$= \binom{3}{2} p^2 (1 - p) + \binom{3}{3} p^3$$

$$= 3 \times 10^{-6} (1 - 10^{-3}) + 10^{-9} \cong 3 \times 10^{-6}$$

Example 2 cont'd:

(b) Consider 5 bits (n = 5).

Define: A = {3 or more bit errors in 5 trials}

$$\Pr[\text{error}] = \Pr[\text{A}] = \Pr[r \geq 3]$$

$$= \binom{5}{3} p^3 (1-p)^2 + \binom{5}{4} p^4 (1-p) + \binom{5}{5} p^5$$

$$\cong 9.985 \times 10^{-9}$$

Example:

Consider a sequence of 10 binary digits. Let Pr[1] = 0.52.

(a) What is the probability of obtaining 8 or more 1's?

Define: A = {8 or more 1's in a sequence of 10 bits}

$$\Pr[A] = \binom{10}{8} \cdot 0.52^8 \cdot 0.48^2 + \binom{10}{9} \cdot 0.52^9 \cdot 0.48 + \binom{10}{10} \cdot 0.52^{10}$$

$$= 45 \cdot 0.52^8 \cdot 0.48^2 + 10 \cdot 0.52^9 \cdot 0.48 + 0.52^{10} \cong 0.0702161458426$$

(b) What is the probability of obtaining exactly six 1's?

Define: A = {exactly six 1's in a sequence of 10 bits}

$$\Pr[A] = \binom{10}{6} \cdot 0.52^6 \cdot 0.48^4 = 210 \cdot 0.52^6 \cdot 0.48^4 \cong 0.220396303407$$

The Geometric Probability Law

Consider a sub-experiment. Let A be the desired event.

Let $\Pr[A] = p, \quad \Pr[A^C] = 1 - p.$

Repeat the sub-experiment until A occurs.

- Suppose that A occurs in the k^{th} trial:

A^C	A^C	A^C	A^C	…	A^C	A
1	2	3	4	…	k-1	k

← Other than desired → Desired

The probability that A occurs in the k^{th} trial is:

$$\Pr[\text{A occurs in } k^{th} \text{ trial}] = \underbrace{(1-p)(1-p)(1-p)(1-p)\cdots(1-p)}_{k-1 \text{ uneventful trials}} \quad p$$

$$= (1-p)^{k-1} \quad p$$

Example:

In a computer to computer modem link, the receiving computer has an error detection algorithm. If it detects a bit error, it requests retransmission of the packet. For simplicity, assume that the packet length is 8 bits. Let the probability of channel error be Pr[error] = 0.1.

(a) Determine the probability that the error occurs after the 5th bit in a packet.

$$\begin{aligned}\Pr[k > 5] &= \Pr[k = 6] + \Pr[k = 7] + \Pr[k = 8] \\ &= 0.9^5 \cdot 0.1 + 0.9^6 \cdot 0.1 + 0.9^7 \cdot 0.1 \\ &= 0.16002279\end{aligned}$$

Example cont'd:

(b) What is the probability that a packet is retransmitted 2 times?

- A packet is retransmitted once if at least one of the 8 bits is in error.
- It is retransmitted again if at least one of the retransmitted 8 bits is in error.

$$\Pr[1 \text{ retransmission}] = \Pr[k \geq 1] = \sum_{i=1}^{8} \Pr[k = i] = 0.5695$$

$$\Pr[2 \text{ retransmissions}] = \left(\Pr[1 \text{ retransmission}]\right)^2 = 0.3244$$